

# The data aggregation problem in quantum hypothesis testing

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**Abstract.** We discuss the implications of quantum-classical Yule-Simpson effect for quantum hypothesis testing in the presence of noise, and provide an experimental demonstration of its occurrence in the problem of discriminating which polarization quantum measurement has been actually performed by a detector box designed to measure linear polarization of single-photon states along a fixed but unknown direction.

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## 1 Introduction

The Yule-Simpson effect [1, 2, 3] in statistics occurs when the correlations observed within different samples are reversed when the sampled are combined together. Although no actual mathematical paradox is involved, the Yule-Simpson effect has an impact on statistical inference, since the aggregated data and the partitioned ones may suggest opposite conclusions. Two forms of the Yule-Simpson effect in quantum measurements has been recently introduced in [5] and their occurrence in qubit systems have been experimental verified [6]. The possible connections of the effect with high order Bell-Tsirelson inequalities have been also explored [7].

In this paper we discuss the implications of quantum-classical Yule-Simpson effect for quantum hypothesis testing in the presence of noise. In particular, we demonstrate its occurrence in the problem of discriminating which polarization quantum measurements has been actually performed by a given *detector box*, designed to measure linear polarization of single-photon states along one of two possible directions.

Suppose that you are given a box, which may implement two possible dichotomic measurements  $A = \{\Pi_A, \mathbb{I} - \Pi_A\}$  and  $B = \{\Pi_B, \mathbb{I} - \Pi_B\}$  on a given system, and you have to infer which measurement has been performed on the basis of the results of the measurement. To this aim, you may *probe* the measuring box  $M$  times by suitably prepared states of the system. In our scheme the box is performing (linear) polarization measurements along a given direction, or along a slightly tilted one. Let us denote by  $\theta$  the possible tilting angle. The two measurements are thus described by the operator measures  $\Pi_A = |0\rangle\langle 0|$ ,  $|0\rangle$  describing vertical polarization in the given direction, or  $\Pi_B = |0\rangle_\theta\langle 0|$ , where  $|0\rangle_\theta = \cos\theta|0\rangle + \sin\theta|1\rangle$  [8].

In order to discriminate which measurement has been actually performed, one sends some probe signal and take

a decision on the basis of the measurement results. As for example, we may send photons with a definite polarization state, e.g.  $\varrho_0 = |0\rangle\langle 0|$ , corresponding to linear vertical polarization along the given direction. In this case the detector always returns the "0" outcome if the box is performing  $A$  measurement, while some fraction of "1" is expected in case of the  $B$  measurement. More precisely, the probabilities of obtaining the outcome "0" with the two measurements are given by

$$\begin{aligned} p_1 &= \langle 0 | \Pi_A | 0 \rangle = 1 \\ q_1 &= \langle 0 | \Pi_B | 0 \rangle = |\langle 0 | 0 \rangle_\theta|^2 = \frac{1}{2}(1 + \cos 2\theta) \end{aligned} \quad (1)$$

Let us now admit that some external perturbation may introduce some noise in the preparation stage of the probe signal. In particular, we assume that if the noise is present then the probe is prepared in a mixture of states having linear vertical polarization along a random direction, tilted by small angle  $\alpha$  from the given axis. In order to make minimal assumptions on the nature of the perturbation, we take the angles  $\alpha$  distributed according to a Gaussian with zero mean. In this case the polarization state of the probing photons is described by the density operator

$$\varrho_\Delta \equiv \mathcal{D}_\Delta(\varrho_0) = \int d\alpha \frac{e^{-\frac{\alpha^2}{2\Delta^2}}}{\sqrt{2\pi\Delta^2}} |0\rangle_{\alpha\alpha}\langle 0|, \quad (2)$$

where  $\Delta \ll 2\pi$ , such that the integral may be safely evaluated over the entire real axis. The probabilities of getting the "0" outcome for the two measurements are now given by

$$\begin{aligned} p_2 &= \langle 0 | \varrho_\Delta | 0 \rangle = \frac{1}{2}(1 + \delta) \\ q_2 &= \langle 0 | \varrho_\Delta | 0 \rangle_\theta = \frac{1}{2}(1 + \delta \cos 2\theta), \end{aligned} \quad (3)$$

where  $\delta = \exp(-2\Delta^2)$  represents the smearing effects of the preparation noise.

At first sight, the presence of preparation noise is not changing the picture. Indeed, we have that  $p_2$  is larger than  $q_2$ , such that one still expects a larger number of "0" outcomes when the box is performing the  $A$  measurement.

On the other hand, and this is the *data aggregation problem* that we mention in the title of the paper, if we do not know how many times the perturbation in the preparation state occurred, it could happen that the overall probability of the event "0" is larger for the  $B$  measurement than for the  $A$  measurement, i.e. we may expect more "0" by measuring polarization along the tilted direction than with the original one. In order to understand how this may happen, let us denote by  $\gamma = M_0/M$  the fraction of runs where the box is probed by the state  $\varrho_0$ . The overall density operator describing the polarization state of the probing photon is given by

$$\varrho_\gamma \equiv \Phi_{\gamma\Delta}(\varrho_0) = \gamma \varrho_0 + (1 - \gamma) \varrho_\Delta \quad (4)$$

which may be seen as the output state from an overall two-parameter noisy channel described by the map

$$\Phi_{\gamma\Delta} = \gamma \mathcal{I} + (1 - \gamma) \mathcal{D}_\Delta, \quad (5)$$

being  $\mathcal{I}$  the identity channel and  $\mathcal{D}_\Delta$  the phase-diffusion one, introduced in Eq. (2).

The probabilities of the "0" outcome for the two measurements is given by

$$\begin{aligned} p &= \langle 0 | \varrho_\gamma | 0 \rangle = \gamma p_1 + (1 - \gamma) p_2 \\ q &= \langle 0 | \varrho_\gamma | 0 \rangle_\theta = \gamma q_1 + (1 - \gamma) q_2. \end{aligned} \quad (6)$$

The data aggregation problem consists in the fact that there exist frequencies  $\gamma_1$  and  $\gamma_2$  such that  $\gamma_2 q_1 + (1 - \gamma_2) q_2 > \gamma_1 p_1 + (1 - \gamma_1) p_2$  despite the fact that  $p_1 > q_1$  and  $p_2 > q_2$ . This happens if

$$\gamma_2 > \frac{p_1 - p_2}{q_1 - q_2} \gamma_1 + \frac{p_2 - q_2}{q_1 - q_2},$$

i.e.

$$\gamma_2 > \frac{\gamma_1}{\cos 2\theta} + \frac{\delta}{1 - \delta} \frac{1 - \cos 2\theta}{\cos 2\theta}.$$

Remarkably, the above relation may be satisfied by some pairs of frequencies  $\gamma_1$  and  $\gamma_2$  whenever  $\delta < 2 \cos 2\theta$ . For fixed frequencies the effect takes place if the preparation noise is larger than a threshold, corresponding to

$$\delta < \frac{\gamma_1 - \gamma_2 \cos 2\theta}{\gamma_1 - 1 - (\gamma_2 - 1) \cos 2\theta} \stackrel{\theta \ll 1}{\simeq} 1 - \frac{2\theta^2}{\gamma_2 - \gamma_1}$$

Summarizing, we probe the detector box by a pair of possible preparations, described by the density operators  $\varrho_0$  and  $\varrho_\Delta$ , corresponding to negligible noise acting on the probe ( $\varrho_0$ ) or to the presence of non-negligible noise described by Gaussian mixing ( $\varrho_\Delta$ ). After the measurement, we aim to infer which polarization has been actually measured on the basis of the number of, say, "0" outcomes recorded after  $M = M_0 + M_\Delta$  repeated measurements,

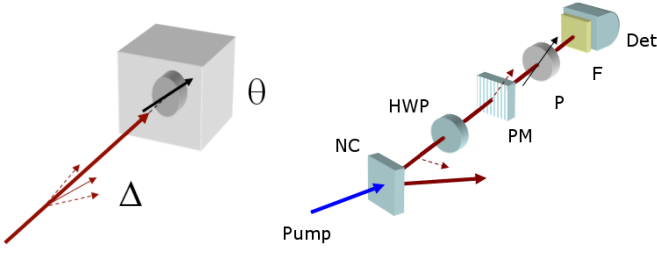
where  $M_j$  is the number of runs where the system was prepared in the state  $\varrho_j$ ,  $j = 0, \Delta$ . If we know which preparation  $\varrho_j$  has been used in each run, i.e. we know when the noise is present, then we are able to make a definite inference, say  $A$  measurement if  $p_j > q_j$ , independently on the number of runs. On the other hand, if we ignore the information about which preparation has been sent to the box in each run, i.e. we aggregate data because we do not know whether the noise was present or not, then we may reach the opposite conclusion, depending on the relative weight  $M_0/M_\Delta$  of the samples. This is a manifestation of the quantum-classical Yule-Simpson effect, which may easily occur when discriminating measurement apparatuses in the presence of noisy channels described by maps of the form (5). Overall, there is no mathematical paradox: still the aggregated data and the partitioned ones may, in fact, suggest opposite conclusions. The effect is referred to as *quantum-classical* YS effect since it occurs in quantum measurements due to classical uncertainty in the preparation of the probe signals, i.e. to the presence of mixed probes. An analogue *quantum-quantum* YS effect may indeed occur with superpositions [5].

In the next Sections we describe and discuss an experimental scheme where the above effect takes place.

## 2 Experimental apparatus

The logical scheme of the experiment, corresponding to the situation described in Section 1, is shown in the left panel of Fig. 1, whereas the experimental setup is shown in the right panel of the same figure. We work with photon polarization since this is a degree of freedom which may be reliably controlled. In turn, it has been already shown that the noise model introduced in the previous section may be reliably implemented [9, 10].

A linearly polarized cw 405 nm diode laser (Newport LQC405-40P) pumps a  $\beta$ -barium borate crystal (NC, length 3 mm) cut for type-I down conversion with the optical axes aligned in the horizontal plane. The non-linear crystal is used as a source of horizontally polarized photon pairs via parametric down conversion. We use an half-wave-plate (HWP) to set the polarization at  $45^\circ$ . Then, in order to obtain a scheme equivalent to that of the left panel of Fig. 1, we use a phase modulator and a polarizer set at  $45^\circ$  (see below). Finally, we have a long-pass filter (cut-on wavelength = 780 nm) to reduce the background and an home-made single photon detector (Det). With the phase modulator it is possible to introduce an arbitrary phase shift  $\phi$  between the horizontal (H) and the vertical (V) polarization. After the polarizer set a  $45^\circ$  the probability to see a photon is thus  $\frac{1}{2}(1 + \cos \phi)$ . The acquisition consist of 200 iterations. For each iteration we acquire 4 counts, each within a temporal window of 1 second:  $N_{1p}$  are the counts obtained for  $\phi = 0$ ,  $N_{1q}$  are for the setting  $\phi = 2\theta$ ,  $N_{2p}$  corresponds to  $\phi = -2\alpha$ , and  $N_{2q}$  to  $\phi = 2(\theta - \alpha)$ , where  $\alpha$  is randomly sampled from a Gaussian distribution of zero mean and variance  $\Delta$ . Since, according to Eq.(1), for  $\phi = 0$  we have  $p_1 = 1$ , then  $N_{1p}$  is used as a normalization



**Fig. 1.** (Color online) (Left): Basic blocks of the experiment. A light beam which *may* be subject to Gaussian polarization diffusion ( $\Delta$  is the standard deviation) enters in a detector box which contains a linear polarization analyzer set at an angle  $\theta$  with respect to a reference axis. (Right): Schematic diagram of the experimental apparatus. A  $\beta$ -barium borate crystal (NC, length 3 mm), pumped by a linearly polarized cw 405 nm diode laser, is the source of horizontally polarized photon pairs via parametric down-conversion. Then the polarization is set at  $45^\circ$  by an half-wave-plate (HWP). The ideal scheme is simulated introducing a proper phase shift by a phase modulator (PM) and a polarizer (P) set a  $45^\circ$ . (F) is a long-pass filter (cut-on wavelength = 780 nm) and (Det) is a single photon detector.

to estimate the other probabilities as follows:

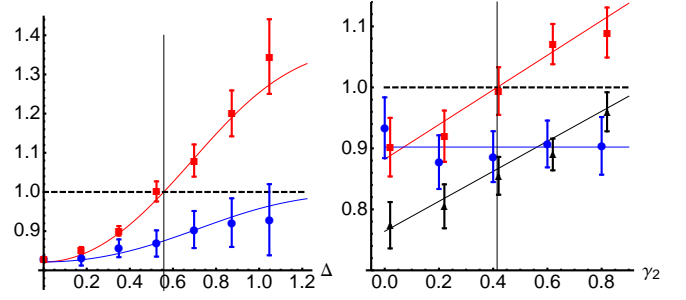
$$q_1 = N_{1q}/N_{1p} \quad p_2 = N_{2p}/N_{1p} \quad q_2 = N_{2q}/N_{1p}.$$

After the acquisition of the four counts, we emulate the lack of knowledge about the preparation of the probe by mixing the  $N_p$  and the  $N_q$  data according to a pair of dichotomic distributions  $(\gamma_1, 1 - \gamma_1)$  and  $(\gamma_2, 1 - \gamma_2)$ . We thus obtain *online* the ratios  $q_1/p_1$  and  $q_2/p_2$ , as well as  $q/p$ , together with their corresponding uncertainties.

### 3 Results

Experimental results are summarized in Fig. 2. In the left panel we show the ratios  $q_2/p_2$  (blue circles) and  $q/p$  (red squares) as a function of the preparation noise parameter  $\Delta$  for fixed values of the frequencies  $\gamma_1 = 0.1$ ,  $\gamma_2 = 0.8$  and for the alternative measurement taken at  $\theta = 25^\circ = \frac{5}{36}\pi$  rad (the ratio  $q_1/p_1$  is smaller than unit by construction). Data are in excellent agreement with the theoretical predictions of Eqs. (3) and (6) (solid lines) and confirm the occurrence of the YS effect in quantum hypothesis testing in the presence of noise. For our choice of  $\gamma_1$ ,  $\gamma_2$  and  $\theta$  the noise threshold for the YS effect was  $\Delta > \Delta_{th} \simeq 0.558$  rad.

In the right panel we show the ratios  $q_2/p_2$  (blue circles) and  $q/p$  (red squares for  $\gamma_1 = 0.05$  and black triangles for  $\gamma_1 = 0.4$ ) as a function of probability  $\gamma_2$  for a fixed value of the preparation noise  $\Delta = \frac{2}{9}\pi$  rad and for the alternative measurement taken at  $\theta = \frac{5}{36}\pi$  rad. Data are in excellent agreement with the theoretical predictions of Eqs. (3) and (6) (solid lines), confirming that  $q_2/p_2$  is independent on the choice of the probabilities  $\gamma_1$  and  $\gamma_2$ , and showing that YS effect may occur for increasing  $\gamma_2$ . The vertical line denotes the threshold for the occurrence



**Fig. 2.** (Color online) (Left): the ratios  $q_2/p_2$  (blue circles) and  $q/p$  (red squares) as a function of the preparation noise parameter  $\Delta$  for fixed values of the frequencies  $\gamma_1 = 0.1$ ,  $\gamma_2 = 0.8$  and for the alternative measurement taken at  $\theta = \frac{5}{36}\pi$  rad. Solid lines denotes the theoretical predictions of Eqs. (3) and (6). The vertical line denotes the noise threshold for the occurrence of the YS effect at the given values of  $\gamma_1$ ,  $\gamma_2$ , and  $\theta$ , i.e.  $\Delta_{th} \simeq 0.558$  rad. (Right): the ratios  $q_2/p_2$  (blue circles) and  $q/p$  (red squares for  $\gamma_1 = 0.05$  and black triangles for  $\gamma_1 = 0.4$ ) as a function of probability  $\gamma_2$  for a fixed value of the preparation noise  $\Delta = \frac{2}{9}\pi$  rad and for the alternative measurement taken at  $\theta = \frac{5}{36}\pi$  rad. Solid lines denotes the theoretical predictions of Eqs. (3) and (6). The vertical line denotes the threshold for the occurrence of the YS effect for  $\gamma_1 = 0.05$  and at the given values of  $\theta$  and  $\Delta$ , i.e.  $\gamma_2 = 0.414$  (no YS effect for  $\gamma_1 = 0.4$ ). Notice that the  $q/p$  data in the right panel have been slightly shifted to the right for clarity but they have been collected for the same values of  $\gamma_2$  as the  $q_2/p_2$  ones.

of the YS effect for  $\gamma_1 = 0.05$  and at the given values of  $\theta$  and  $\Delta$ , i.e.  $\gamma_2 = 0.414$ , whereas, as expected, no YS effect occurs for  $\gamma_1 = 0.4$ .

### 4 Conclusions

In this paper, we have discussed the implications of quantum-classical Yule-Simpson effect for quantum hypothesis testing and demonstrated its occurrence in the problem of discriminating which polarization quantum measurements has been actually performed by a given box, with the two possible detectors designed to measure linear polarization of single-photon states along slightly different directions. If noise affects the preparation stage, one is actually probing the box with two different kinds of signals, the unperturbed one and its noisy version. Since one usually ignores which preparation actually arrived at the detector in each run, data from the two preparations are *aggregated* and one may reach opposite inference, depending on the noise occurrence rate. This is a plain manifestation of the quantum-classical Yule-Simpson effect, which may easily occur when discriminating measurement apparatuses in the presence of noise. Overall, there is no mathematical paradox: still the effect is puzzling for what concerns statistical inference, since the aggregated data and the partitioned ones may, in fact, suggest opposite conclusions.

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